where $\beta_2 > \beta_1$ and $\rho_0 > \rho_i$.

Evaluation of Eq. (8) in closed form is at best difficult. Therefore, Eq. (8) was evaluated numerically, and some typical results are shown in Fig. 3. Note that at $\theta_2 = 360°$ (θ_1 assumed to equal zero), the value of $F_{A_1-A_2}$ is the same as that for two fully circular disks regardless of the values of β_1 and β_2. Also note that for $\beta_2 - \beta_1 = 360°$, $F_{A_1-A_2}$ is a straight line.

References

[1] Hamilton, D.C. and Morgan, W.R., "Radiant-Interchange Configuration Factors," NACA TN 2836, 1952, pp. 37, 42.

[2] Sparrow, E.M., "A New and Simpler Formulation for Radiative Angle Factors," *Transactions of the ASME, Series C: Journal of Heat Transfer,* Vol. 85, 1963, pp. 81-88.

[3] Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products,* 4th ed, Academic Press, New York, 1965, pp. 68, 148.

Finite Contact of an Inflated Ring

D. W. Nicholson*
Goodyear Research, Akron, Ohio

Introduction

THE numerical solution is given for a ring which is a) inflated, b) uniform and homogeneous, c) initially circular, and d) under finite (large) compression between rigid, flat, parallel plates (see Fig. 1). The uninflated ring and the inflated cylindrical membrane have been treated.[1,2] For combined inflation and bending, an integral was derived in Ref. 3 with limits to be found from several simultaneous nonlinear algebraic equations. Numerical results were reported only for small deviations from the uninflated and the membrane cases.

In the present full numerical solution, the transition from bending-dominated behavior at low pressure to membrane-dominated behavior at high pressure is studied. Also, we assess an approximate solution[2] in which the force at a given deflection is assumed to consist of independent bending and membrane contributions.

Analysis

1. Nomenclature and Governing Equations

Let X and Y denote the extrinsic coordinates of a point on the ring; denote the pressure as P (positive outward), and let M, V, and T, respectively, denote bending moment (positive anticlockwise), shear force (positive outward), and tension (positive clockwise). Also, R_0 is the initial radius of the ring. Owing to symmetry, attention will be confined to the first quadrant. Also, a unit width in the axial direction is assumed.

The analysis will be in terms of intrinsic coordinates ϕ and s, defined such that $\tan\phi = dY/dX$ where $dX = ds \cos\phi$. As an elastica the ring obeys

$$M = -D[(d\phi/ds) - (d\phi_0/ds)] \tag{1}$$

where the zero subscript refers to the original configuration, and D is the bending rigidity (assumed constant). The quantity $(d\phi/ds)^{-1}$ is the radius of curvature.

We introduce the dimensionless quantities $x = X/R_0$, $y = Y/R_0$, $\sigma = S/R_0$, $m = MR_0/D$, $v = VR_0^2/D$, $t = TR_0^2/D$, and $p = PR_0^2/D$. The required geometric and equilibrium relations are[1]

$$dx/d\sigma = \cos\phi \tag{2a}$$

Received Oct. 20, 1975; revision received March 11, 1976.
Index categories: Structural Static Analysis.
*Senior Physicist.

$$dy/d\sigma = \sin\phi \tag{2b}$$

$$dv/d\sigma = p - tz \tag{2c}$$

$$dt/d\sigma = vz \tag{2d}$$

$$dz/d\sigma = -V \tag{2e}$$

where z is defined by

$$d\phi/d\sigma = z \tag{2f}$$

2. Boundary Conditions and Solution Method

For loads below a critical value, say F^*, the initial contact point and its neighboring points have infinite radii and belong to the now finite contact zone. Boundary conditions are given below.

For point contact, at $\sigma = 0$,

$$\phi = 0, \quad z = z_0 \tag{3a}$$

where z_0 is specified. At $\sigma = \pi/2$

$$\phi = \frac{\pi}{2}, \quad v = 0, \quad y = 0 \tag{3b}$$

It remains to determine v and y at $\sigma = 0$. Denote these unknowns as $\bar{f}/2$ and $L = \bar{L}/R_0$. If these quantities were known, the foregoing would reduce to an initial value problem with initial conditions at $\sigma = 0$

$$\phi = 0, \quad z = z_0, \quad v = \bar{f}/2, \quad t = pL, \quad y = L \tag{4}$$

For finite contact, denoting $s_1 = R_0\sigma_1$ as the contact half-width, at $\sigma = \sigma_1$

$$\phi = 0, \quad z = 0 \tag{5a}$$

at $\sigma = \pi/2$

$$\phi = \pi/2, \quad v = 0, \quad y = 0 \tag{5b}$$

In finite contact, the total (dimensionless) compressing force on the ring is

$$f = f_b + 2p\,\sigma_1 \tag{6a}$$

where f_b is a concentrated force, interpreted as the bending contribution, acting at σ_1. For point contact, the total force is

$$f = \bar{f} \tag{6b}$$

and hereafter the overbar will not be displayed.

For brevity we discuss only the numerical solution for the point contact stage. Equation (2) together with boundary conditions Eq. (3) defines a two-point boundary value problem which we have solved by the widely studied "shooting" techniques.[4] The auxiliary initial value problem [involving Eq. (4)] was integrated using Hamming's method, and the assumed values of $\bar{f}$ and L were adjusted iteratively using Newton's method, in order to accommodate all of the boundary conditions. An extrapolation procedure was used

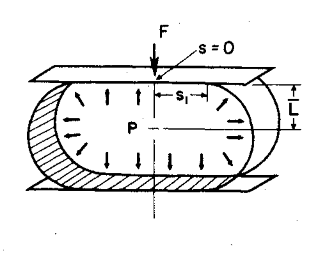

Fig. 1 Inflated ring under compression.

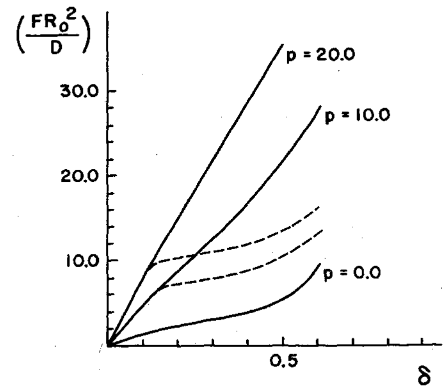

Fig. 2 Force-deflection relation.

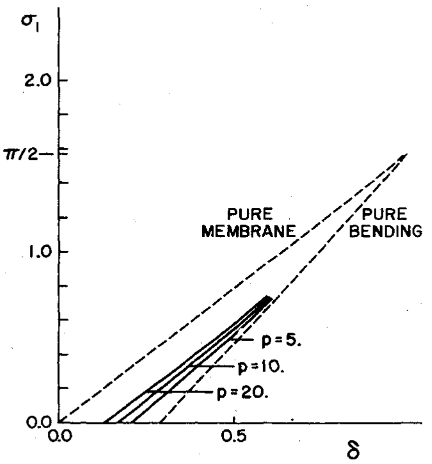

Fig. 3 Contact width against deflection.

together with incrementation to obtain good starting estimates for $\bar{f}$ and L. The accuracy was checked by comparison with the known exact solution[1] for $p=0$ and by comparing the computed value of ϕ at $\sigma=\pi/2$ with $\pi/2$. For proper choices of program parameters, the error as measured by $[\phi(\pi/2)-\pi/2]$ was of order 10^{-5}.

3. Extreme Cases and Approximate Solution

For pure bending ($P=0$) and for pure membrane behavior ($D=0$), the contact half-width σ_l is given by the relations[2]:

Membrane

$$\sigma_l=\sigma_m=\pi\delta/2 \tag{7a}$$

Bending

$$\sigma_l=\sigma_b=(J_l/J_2)(\delta-\delta^*) \quad \text{for } \delta>\delta^* \tag{7b}$$

where $\delta=1-L$ is the deflection, $\delta^*=0.28$, $J_l=2.66206$, and $J_2=1.19814$. In the combined case one expects that $\sigma_m>\sigma_l>\sigma_b$ and $\sigma_l\rightarrow\sigma_m$ as $p\rightarrow\infty$. How rapidly σ_l approaches σ_m with increasing pressure is a measure of the rate of transition from bending-dominated to membrane-dominated behavior.

In the limiting cases the force-deflection relations are[2]:

Membrane

$$F=\pi PR_0\delta \tag{8a}$$

Bending

$$F=DJ_2^2/[R_0^2(1-\delta)^2] \tag{8b}$$

It was argued in Ref. 2 that the force at a given deflection in the combined case should be well approximated by the simple

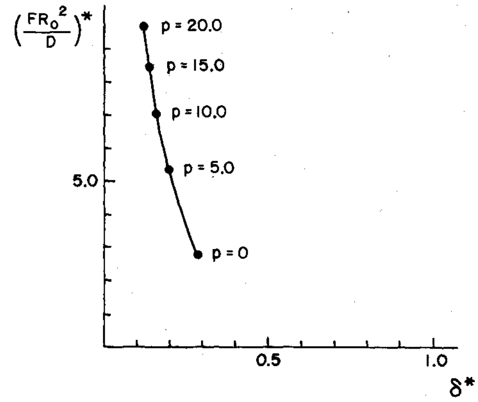

Fig. 4 Effect of pressure on transition from point to finite contact.

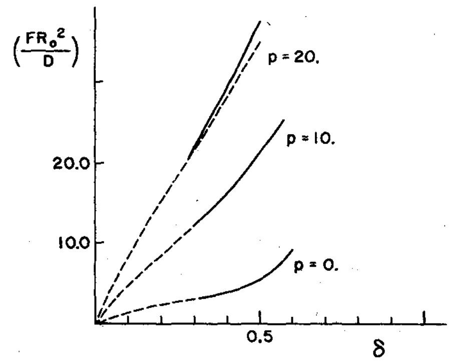

Fig. 5 Comparison of exact and approximate solutions.

sum of the corresponding forces in the limiting cases:

$$f=\pi p\delta+J_2^2/(1-\delta)^2 \quad \text{for } \delta>\delta^* \tag{8c}$$

We assess this approximation in the next section.

4. Results and Conclusions

Calculations for $p=0$, 10, and 20 furnish several conclusions:

1) Figure 2 gives the force-deflection curves, the dashed lines being the bending contribution during finite contact. Obviously, the bending contribution is affected by pressure, i.e. bending and membrane affects are coupled.

2) In Fig. 3 contact half-width is seen to lie between the limiting cases. Even at $p=20$, σ_l has not approached σ_m very closely. The transition to membrane behavior is very gradual.

3) Figure 4 gives the critical load and deflection for the transition from one point to finite contact. As Δ^* approaches zero, F^* grows very rapidly, underscoring the gradual character of the transition to membrane behavior.

4) The dashed lines of Fig. 5 represent the exact (computed) solution whereas the solid lines are the approximate solution (Eq. 8c). Despite the strong coupling of membrane and bending effects, the approximation is very good.

5) In addition, at a given δ, f depends linearly on the parameter p, which can be interpreted as a ratio of membrane to bending forces.

References

[1]Frisch-Fay, R., *Flexible Bars*, Butterworth Co., Ltd., London, 1962.

[2]Nicholson, D. W., "A Model Analysis of the Structural and Pneumatic Contributions to Tire Behavior Under Vertical Loads," *Tire Science and Technology*, Vol. 3, Feb. 1975, pp. 29-42.

[3]Martin, F., "Theoretische Untersuchungen zur Frage des Spannungszustandes in Luftreifen bei Abplattung," *Jahrbuch der Deutschen Luftfahrtforschung*, Pt. 1, 1939, pp. 470-478.

[4]Keller, H. B., *Numerical Methods for Two-Point Boundary Value Problems*, Blaisdell Publishing Co., London, 1968.